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A NEW GENERAL THEORY OF ERRORS.

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I. Introduction.

The ordinary theory of errors — the method of least squares — has been developed in three ways (methods of Gauss, Hagen, and Crofton); not only is each of these developments based upon assumptions to which not every one will assent, but at certain stages in each of the two later developments exactness is replaced by approximation through the neglect of certain quantities (or terms) regarded as very small (or productive of very little effect) in comparison with others retained, whereas this neglect is not fully justified, as it seems to us.

The present paper is the outcome of an attempt to construct a theory of errors upon such simple principles as will be generally admitted to be necessary for the mathematical treatment of any such theory. The immediate object of the theory of errors is to determine the probability that the error made in the direct measurement of a physical magnitude shall lie between given limits. More complex applications of the theory (e. g. to functions of quantities directly observed and to quantities determined by calculations based upon direct observations) can be connected with the solution of the fundamental problem just stated by means familiar to all who have used the ordinary theory. Of course we consider only accidental errors, assuming the observations to have been corrected for all systematic errors before being subjected to our investigation; we suppose also that the observations were all made under the same conditions, so that they are equally reliable (their relative probabilities must be determined by the theory).

II. Case of a Finite Number of Observations.

Let $z_1, z_2, z_3, \ldots, z_n$ be n observed values of a magnitude whose true value is Z, and let z denote an observed value in general. If A is any function of z and $A_1, A_2, A_3, \ldots, A_n$ are the values of this function for $z_1, z_2, z_3, \ldots, z_n$, respectively, we write

$$A_1 + A_2 + A_3 + \ldots + A_n = [A],$$

as is customary in the theory of errors. Let z_0 be the arithmetical mean of the observed values, so that

$$(1) \qquad z_0 = \frac{1}{n}[z].$$

Let x be the error of an observation in general, $x_1, x_2, x_3, \ldots, x_n$ the errors of the n given observations, and x_0 the arithmetical mean of these errors, so that

$$(2) \quad \begin{cases} x = z - Z, x_1 = z_1 - Z, x_2 = z_2 - Z, x_3 = z_3 - Z, \ldots, x_n = z_n - Z, \\ x_0 = \frac{1}{n}[x] = \frac{1}{n}[z] - Z = z_0 - Z; \end{cases}$$

therefore, the arithmetical mean of the errors is the error of the arithmetical mean of the observations. Let ξ be the *residual* of an observation in general, — that is, the excess of the observed value over the arithmetical mean of the given observed values, — and let $\xi_1, \xi_2, \xi_3, \ldots, \xi_n$ be the residuals of the several observations, so that

$$(3) \qquad \xi = z - z_0 = (x + Z) - (x_0 + Z) = x - x_0;$$

therefore, the residual of an observed value is the residual of its error; we may call it simply the *residual* of the observation. Let p_k be the mean k-th power of the residual, so that

$$(4) \qquad p_k = \frac{1}{n}[\xi^k];$$

in particular,

$$(5) \quad \begin{cases} p_0 = \frac{1}{n}[\xi^0] = \frac{1}{n}[1] = 1, \\ p_1 = \frac{1}{n}[\xi] = \frac{1}{n}[z] - z_0 = 0. \end{cases}$$

The positive square root of the mean square of the residual shall be called the *mean residual* (it is called the *mean error* in the ordinary theory) and shall be represented by μ, so that

$$(6) \qquad \mu = \sqrt{p_2} \quad \text{or} \quad p_2 = \mu^2.$$

The ratio of the residual to the mean residual shall be called the *relative residual* and be represented by s, while the ratio of the mean k-th power of the residual to the k-th power of the mean residual shall be called the *relative mean* k-*th power* of the residual and be represented by ρ_k, so that

$$
\text{(7)} \qquad \begin{cases} s = \dfrac{\xi}{\mu}, \; \rho_k = \dfrac{p_k}{\mu^k}, \\[2ex] \rho_0 = \dfrac{p_0}{\mu^0} = 1, \; \rho_1 = \dfrac{p_1}{\mu} = 0, \; \rho_2 = \dfrac{p_2}{\mu^2} = 1. \end{cases}
$$

A little consideration will make it clear that no observations, however many, can determine the *true* value of the observed magnitude, nor, therefore, the *errors* of the several observations, — they can determine only what is equivalent to their arithmetical mean and the residuals of the observations. The fundamental problem of the theory of errors must, therefore, be modified, and may now be stated to be the determination of the probability that the residual of an observation shall lie between certain limits according to the *law of the long run*, that is, in general, on the assumption that the distribution of errors over the whole range of possible error tends to follow a certain law, as the number of observations is increased without limit, and that the *proportion* of all the errors of *any* (finite) number of observations that fall between given limits is the same as if the number of observations was infinite. It is evident that the distribution of errors among different intervals in definite proportions implies an increase in the breadth of the intervals as the number of observations is diminished. The analytical theory of errors is based upon the supposition that the number of observations is infinite and that the interval for which the proportion of errors is to be determined may be taken to be as narrow as we please.

In connection with this statement of the fundamental problem it may be said that the determination of the value that may *best* be assigned to the observed magnitude is not explicitly a part of the fundamental problem, although immediately connected with it and perhaps the most desirable result of it from the practical point of view, but the criterion for the best value is by no means evident. Perhaps it will be generally considered that the best value is the *probable value*, that is, the value that is just as likely to be exceeded as not in the long run, so that the probability that the observed value shall fall short of the probable value is $\frac{1}{2}$ and the probability that the observed value shall exceed the probable value is $\frac{1}{2}$. Certainly the best value cannot be defined as the *most* probable value, that is, the value whose probability is a maximum, because it turns out that the probability is sometimes a maximum for more than one value; the probability may even have equal maxima for two or more values.

III. CASE OF AN INFINITE NUMBER OF OBSERVATIONS — ASSUMPTIONS.

The assumptions upon which we shall base the general theory of errors (upon which the ordinary theory has always been based in part, explicitly or implicitly) are these:

I. *Possible errors form a practically continuous sequence from a certain lower limit to a certain upper limit;*

II. *The probability that the error of an observation lies between* x *and* $x + dx$*, where* dx *is infinitesimal, is* $\phi(x) \cdot dx$*, where* $\phi(x)$ *is an analytical function of* x *developable by Taylor's theorem throughout the whole range of possible error;*

III. *The probability that the error lies between given limits is independent of the unit of measurement.*

Assumption I forms the foundation for II, and II implies that $\phi(x + h)$ is developable according to powers of h by Taylor's theorem for all values of x and $x + h$ that lie within the range of possible error. The value of $\phi(x)$ for any given value of x is called the *relative frequency* of the error x.

The probability that the error of an observation lies between two given limits a (lower) and b (upper) is, evidently,

$$\int_a^b \phi(x) \cdot dx. \tag{8}$$

If $\underline{x}$ is the (algebraically) least possible error and $\bar{x}$ the greatest possible error, we have

$$\int_{\underline{x}}^{\bar{x}} \phi(x) \cdot dx = 1. \tag{9}$$

If we express x in terms of the residual ξ by (3), representing the lower and upper limits of possible residuals by $\underline{\xi}$ and $\bar{\xi}$, respectively (which correspond to the extreme errors $\underline{x}$ and $\bar{x}$), and writing

$$\phi(x) = \phi(\xi + x_0) = \psi(\xi), \tag{10}$$

we have from (9), because x_0 is constant,

$$\int_{\underline{\xi}}^{\bar{\xi}} \psi(\xi) \cdot d\xi = 1, \tag{11}$$

and the probability that the residual of an observation lies between a and b is

$$(12) \qquad \int_a^b \psi(\xi) \cdot d\xi.$$

The value of $\psi(\xi)$ for any given value ξ may be called the *relative frequency* of the residual ξ, and $\psi(\xi) \cdot d\xi$ is evidently the probability that the residual lies between ξ and $\xi + d\xi$; that is, $\psi(\xi) \cdot d\xi$ is the *proportion* of all the residuals that lie between ξ and $\xi + d\xi$. In calculating the mean powers of the residual by (4), the power of each residual is to be multiplied by the number of times it occurs, and the sum of the products thus formed for all the residuals is to be divided by the number of observations; if we assign all the residuals between ξ and $\xi + d\xi$ to the value ξ itself (or replace them all by their mean), the multiplier corresponding to this value is proportional to $\psi(\xi) \cdot d\xi$, while the whole number of observations is proportional to (11). We have then

$$(13) \qquad p_k = \int_{\underline{\xi}}^{\bar{\xi}} \xi^k \psi(\xi) \cdot d\xi,$$

and, in particular, by (5) and (6),

$$(14) \qquad \int_{\underline{\xi}}^{\bar{\xi}} \psi(\xi) \cdot d\xi = p_0 = 1,$$

$$(15) \qquad \int_{\underline{\xi}}^{\bar{\xi}} \xi \psi(\xi) \cdot d\xi = p_1 = 0,$$

$$(16) \qquad \int_{\underline{\xi}}^{\bar{\xi}} \xi^2 \psi(\xi) \cdot d\xi = p_2 = \mu^2.$$

Therefore, by (3), (14), and (15),

$$(17) \qquad \int_{\underline{x}}^{\bar{x}} x \cdot \phi(x) \cdot dx = \int_{\underline{\xi}}^{\bar{\xi}} (\xi + x_0) \cdot \psi(\xi) \cdot d\xi = x_0$$

Because $\psi(\xi)$ is essentially positive for any possible residual ξ, formula (15) shows that the possible values of ξ must include both positive and negative values, so that $\underline{\xi} < 0$, $0 < \bar{\xi}$, and 0 is a possible residual in all cases.

IV. Effect of an Infinitesimal Source of Error.

To determine the form of the function $\psi(\xi)$ we consider the effect of the addition of an *infinitesimal* source of error to the original source (however complex that may be), that is, of a source of error that produces infinitesimal errors with great frequency and finite errors of even moderate magnitude with infinitesimal frequency. We employ the same notation for all functions and quantities pertaining to the action of the infinitesimal source alone as for those pertaining to the original source alone, distinguishing them by an accent, and denote changes due to the addition of the infinitesimal source by the symbols for the functions and quantities changed with a δ prefixed.

A given error x produced by the simultaneous action of both sources is the result of a certain error x' produced by the infinitesimal source and an error $x - x'$ produced by the original source. The probability that the error produced by the combination shall lie between x and $x + dx$ is, then,

$$[\phi(x) + \delta\phi(x)]\, dx = \int_{\underline{x}'}^{\overline{x}'} \phi(x - x') \cdot \phi'(x') \cdot dx \cdot dx',$$

from which follows

$$\phi(x) + \delta\phi(x) = \int_{\underline{x}'}^{\overline{x}'} \phi(x - x') \cdot \phi'(x') \cdot dx'. \tag{18}$$

For the arithmetical mean of the error produced by the combination we have

$$\begin{aligned} x_0 + \delta x_0 &= \int_{\underline{x}+\delta\underline{x}}^{\overline{x}+\delta\overline{x}} x \cdot dx \cdot \int_{\underline{x}'}^{\overline{x}'} \phi(x - x') \cdot \phi'(x') \cdot dx' \\ &= \int_{\underline{x}}^{\overline{x}} x \cdot dx \cdot \int_{\underline{x}'}^{\overline{x}'} \phi(x - x') \cdot \phi'(x') \cdot dx' + \overline{x} \cdot \phi(\overline{x}) \cdot \delta\overline{x} - \underline{x} \cdot \phi(\underline{x}) \cdot \delta\underline{x}. \end{aligned}$$

It is to be observed that, in whatever manner these integrals are evaluated, $x - x'$ must be taken over the whole range of values from $\underline{x}$ to $\overline{x}$. The limits of possible error ($\underline{x}$ and $\overline{x}$) cannot be fixed *a priori*; some may think that large errors or residuals are impossible by the nature of accurate observation, but we prefer to regard the limits as determined by the frequency function $\phi(x)$, namely, we assume that the value of

$\phi(x)$ (which is positive for any possible error x) is 0 at each limit;* if $\phi(x)$ is not 0 for any *finite* value of x, all values of the error are to be regarded as possible. The results will show that $\phi(x) = 0$ for $x = \pm \infty$, and, in general, $\phi(x)$ has a very small value for any large value of x. We have, then,

$$\phi(\bar{x}) = 0 \quad \text{and} \quad \phi(\underline{x}) = 0, \tag{19}$$

so that the last two terms of $x_0 + \delta x_0$ vanish, and, on reversing the order of integration, by (17) and (9),

$$\begin{aligned} x_0 + \delta x_0 &= \int_{\underline{x}'}^{\bar{x}'} \phi'(x') \cdot dx' \cdot \int_{\underline{x}+x'}^{\bar{x}+x'} x \cdot \phi(x - x') \cdot dx \\ &= \int_{\underline{x}'}^{\bar{x}'} \phi'(x') \cdot dx' \cdot \int_{\underline{x}}^{\bar{x}} (x + x') \cdot \phi(x) \cdot dx \\ &= \int_{\underline{x}'}^{\bar{x}'} (x_0 + x')\, \phi'(x')\, dx' = x_0 + x_0'; \end{aligned}$$

so that

$$\delta x_0 = x_0'. \tag{20}$$

By (10), as applied to both component sources and their combination, we have

$$\begin{aligned} &\phi'(x') = \phi'(\xi' + x_0') = \psi'(\xi'), \\ &\phi(x - x') = \phi(\xi - \xi' + x_0 - x_0') = \psi(\xi - \xi' - x_0'), \\ &\phi(x) + \delta\phi(x) = (\phi + \delta\phi)(\xi + x_0) = (\psi + \delta\psi)(\xi - x_0') = \\ &\qquad\qquad \psi(\xi - x_0') + \delta\psi(\xi - x_0'), \end{aligned}$$

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* The relative frequency $\psi(\xi)$ of a *possible* residual ξ must be positive; if ξ varies continuously from zero, either through positive or through negative values, the limit $\bar{\xi}$ or $\underline{\xi}$ must be reached before $\psi(\xi)$ becomes negative; but if $\psi(\xi)$ passes through zero without changing sign as ξ passes through a certain value, — a value that corresponds to a point of contact of the curve represented by $y = \psi(\xi)$ with the axis of abscissae, — that value of ξ is not *necessarily* a limit to the possible values; such a value of ξ could never occur as a residual, although values on both sides of it might occur. However, if the conditions under which the observations are made change continuously in such a manner that the curve $y = \psi(\xi)$ assumes a form in which it touches the axis of abscissae, this must be the limiting (extreme) form, provided a continuation of the change would produce a curve having negative ordinates in the neighborhood of the point of contact; namely, on account of the permanence of the relation (14), the limits $\underline{\xi}$ and $\bar{\xi}$ cannot change discontinuously.

for any *given* value of ξ, because the arithmetical mean of the error corresponding to the function $\phi + \delta\phi$ is $x_0 + \delta x_0 = x_0 + x_0'$. Therefore, (18) becomes

$$\psi(\xi - x_0') + \delta\psi(\xi - x_0') = \int_{\underline{\xi}}^{\overline{\xi}} \psi(\xi - \xi' - x_0') \cdot \psi'(\xi') \cdot d\xi',$$

or, when ξ is replaced by $\xi + x_0'$,

$$\psi(\xi) + \delta\psi(\xi) = \int_{\underline{\xi}'}^{\overline{\xi}'} \psi(\xi - \xi') \cdot \psi'(\xi') \cdot d\xi' \tag{21}$$

for any possible value of ξ. Now, by assumption II, $\phi(x)$ is developable by Taylor's theorem and, therefore, so is $\psi(\xi)$, that is

$$\psi(\xi - \xi') = \sum_0^{\infty}{}_k (-1)^k \frac{\xi'^k}{k!} \frac{\partial^k \psi(\xi)}{\partial \xi^k};$$

therefore, by (13), (14), and (15),

$$\begin{aligned}\psi(\xi) + \delta\psi(\xi) &= \sum_0^{\infty}{}_k (-1)^k \frac{1}{k!} \frac{\partial^k \psi(\xi)}{\partial \xi^k} \int_{\underline{\xi}'}^{\overline{\xi}'} \xi'^k \psi'(\xi') \cdot d\xi' \\ &= \sum_0^{\infty}{}_k (-1)^k \frac{p_k'}{k!} \frac{\partial^k \psi(\xi)}{\partial \xi^k} \\ &= \psi(\xi) + \sum_2^{\infty}{}_k (-1)^k \frac{p_k'}{k!} \frac{\partial^k \psi(\xi)}{\partial \xi^k};\end{aligned}$$

so that

$$\delta\psi(\xi) = \sum_2^{\infty}{}_k (-1)^k \frac{p_k'}{k!} \frac{\partial^k \psi(\xi)}{\partial \xi^k}. \tag{22}$$

Also, by (13) and (21),

$$p_m + \delta p_m = \int_{\underline{\xi}+\delta\underline{\xi}}^{\overline{\xi}+\delta\overline{\xi}} \xi^m \cdot d\xi \cdot \int_{\underline{\xi}'}^{\overline{\xi}'} \psi(\xi - \xi') \cdot \psi'(\xi') \cdot d\xi'$$

$$= \int_{\underline{\xi}}^{\overline{\xi}} \xi^m \cdot d\xi \cdot \int_{\underline{\xi}'}^{\overline{\xi}'} \psi(\xi - \xi') \cdot \psi'(\xi') \cdot d\xi' + \overline{\xi}^m \cdot \psi(\overline{\xi}) \cdot \delta\overline{\xi} - \underline{\xi}^m \cdot \psi(\underline{\xi}) \cdot \delta\underline{\xi}.$$

Reversing the order of integration, observing that $\xi - \xi'$ must take all values from $\underline{\xi}$ to $\bar{\xi}$ and that, by (19), $\psi(\bar{\xi}) = 0$ and $\psi(\underline{\xi}) = 0$, we have

$$p_m + \delta p_m = \int_{\underline{\xi}'}^{\bar{\xi}'} \psi'(\xi') \cdot d\xi' \cdot \int_{\underline{\xi}+\xi'}^{\bar{\xi}+\xi'} \xi^m \cdot \psi(\xi - \xi') \cdot d\xi$$

$$= \int_{\underline{\xi}'}^{\bar{\xi}'} \psi'(\xi') \cdot d\xi' \cdot \int_{\underline{\xi}}^{\bar{\xi}} (\xi + \xi')^m \cdot \psi(\xi) \cdot d\xi$$

$$= \sum_0^m{}_k \binom{m}{k} \int_{\underline{\xi}'}^{\bar{\xi}'} \xi'^k \cdot \psi'(\xi') \cdot d\xi' \cdot \int_{\underline{\xi}}^{\bar{\xi}} \xi^{m\ k} \cdot \psi(\xi) \cdot d\xi$$

$$= \sum_0^m{}_k \binom{m}{k} p_k' \, p_{m-k} = p_m + \sum_2^m{}_k \binom{m}{k} p_k' \, p_{m-k},$$

where $\binom{m}{k}$ denotes the coefficient of x^k in the development of $(1 + x)^m$ by the binomial theorem, that is

$$(23) \qquad \binom{m}{k} = \frac{m\,!}{(m-k)\,!\,k\,!}\,.$$

Therefore,

$$(24) \qquad \delta p_m = \sum_2^m{}_k \binom{m}{k} p_k' \, p_{m-k} \text{ for } 2 \leqq m.$$

In particular, for the smaller values of m,

$$\begin{aligned} &\delta p_0 = 0, \quad \delta p_1 = 0 \quad (\text{because } p_0 = 1 \quad \text{and} \quad p_1 = 0), \\ &\delta p_2 = p_2', \quad \delta p_3 = p_3', \quad \delta p_4 = 6\,p_2 p_2' + p_4', \\ &\delta p_5 = 10\,p_3 p_2' + 10\,p_2 p_3' + p_5', \\ &\delta p_6 = 15\,p_4 p_2' + 20\,p_3 p_3' + 15\,p_2 p_4' + p_6', \quad \text{etc.,} \end{aligned}$$

and, therefore,

$$\begin{aligned} p_2' &= \delta p_2\,, \quad p_8' = \delta p_3\,, \quad p_4' = \delta p_4 - 6\,p_2 \cdot \delta p_2\,, \\ p_5' &= \delta p_5 - 10\,p_3 \cdot \delta p_2 - 10\,p_2 \cdot \delta p_3\,, \\ p_6' &= \delta p_6 - 15\,p_4 \cdot \delta p_2 - 20\,p_3 \cdot \delta p_3 - 15\,p_2\,(\delta p_4 - 6\,p_2 \cdot \delta p_2) \\ &= \delta p_6 - 15\,p_2 \cdot \delta p_4 - 20\,p_3 \cdot \delta p_8 - 15\,(p_4 - 6\,p_2^2)\,\delta p_2, \text{ etc.} \end{aligned}$$

Because the right member of (24) is linear in the p''s and p_m' has the coefficient 1 in the expression for δp_m but does not occur in the expression of any δp with suffix less than m, p_m' can be expressed as a linear function of the δp's with suffixes not greater than m whose coefficients are polynomials in the p's. The expression for p_6' in terms of δp_2, δp_3, δp_4, and δp_6 shows that the coefficients of the δp's in the expressions of the p''s are not generally linear. If we substitute the expressions of the p''s as linear functions of the δp's in (22) we obtain $\delta \psi(\xi)$ as a linear function of the δp's whose coefficients are polynomials in the p's multiplied by derivatives of $\psi(\xi)$ with respect to ξ; therefore, *$\psi(\xi)$ is a function of ξ and the* p*'s with suffixes as great as 2 alone*, say *

$$\psi(\xi) = F(\xi, p_2, p_3, p_4, p_5, \ldots). \tag{25}$$

This result confirms the statement made above that all theory can determine is the probability that the *residual* shall lie between given limits.

V. CHANGE OF THE UNIT OF MEASUREMENT.

If we reduce the unit of measurement in the ratio $1 : \lambda$, the numerical values of all measurements are multiplied by λ, so that, for given observations, x, x_0, ξ, $d\xi$, p_k (for $2 \leqq k$) have to be replaced by λx, λx_0, $\lambda \xi$, $\lambda \cdot d\xi$, $\lambda^k p_k$, and, by assumption III, the probability that the residual lies between $\lambda \xi$ and $\lambda \xi + \lambda \cdot d\xi$ with the new unit is the same as that it lies between ξ and $\xi + d\xi$ with the old unit; therefore,

$$F(\lambda \xi, \lambda^2 p_2, \lambda^3 p_3, \lambda^4 p_4, \ldots) \cdot \lambda\, d\xi = F(\xi, p_2, p_3, p_4, \ldots) \cdot d\xi,$$

that is

$$F(\lambda \xi, \lambda^2 p_2, \lambda^3 p_3, \lambda^4 p_4, \ldots) = \frac{1}{\lambda} F(\xi, p_2, p_3, p_4, \ldots) \tag{26}$$

for all values of ξ, p_2, p_3, p_4, ... The relations (25) and (26) express that $\psi(\xi)$ is a homogeneous function of ξ, $\sqrt{p_2}$, $\sqrt[3]{p_3}$, $\sqrt[4]{p_4}$, etc. of degree -1, that is, with the notation of (6) and (7),

$$\psi(\xi) = \frac{1}{\mu} f(s, \rho_3, \rho_4, \rho_5, \ldots); \tag{27}$$

for convenience, we shall sometimes abbreviate $\psi(\xi)$ to ψ and

$f(s, \rho_3, \rho_4, \rho_5, \ldots)$ to $f(s)$ or f.

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* This conclusion implies that every infinitesimal change in $\psi(\xi)$, for a given value of ξ, can be effected by the addition of an infinitesimal source of error, — which ought, perhaps, to be stated as a fourth assumption lying at the base of the general theory.

VI. DIFFERENTIAL EQUATION OF $f(s)$.

Formulae (22) and (24) give the variations of $\psi(\xi)$ and p_m due to the addition of *any* infinitesimal source of error; the mean square p_2' of the residual due to such a source is an essentially positive infinitesimal; in determining the mean powers of the residual for such a source by (13) we may confine our attention to values of the residual that lie within very narrow limits which, in accordance with the remark made in connection with (15), include 0, so that we need consider only very small values of ξ; the terms of the integral (13) for greater values of k are infinitesimals of a higher order than the corresponding terms of the integral for smaller values of k, so that the integral p_k' for any value of k greater than 2 is an infinitesimal of a higher order than p_2'. We specialize the infinitesimal source of error by assuming $p_k' = 0$ for $3 \overline{\gtrless} k$.* We have then, by (24),

$$\delta p_m = \binom{m}{2} p_{m-2} \cdot p_2' \quad \text{for } 2 \overline{\gtrless} m,$$

from which, together with (6), follows

$$p_2' = \delta p_2 = 2\,\mu \cdot \delta\mu,$$

so that
$$\delta p_m = m\,(m-1) \cdot p_{m-2}\,\mu \cdot \delta\mu,$$

while (22) gives

$$\delta\psi(\xi) = \tfrac{1}{2} p_2' \cdot \frac{\partial^2\psi(\xi)}{\partial\xi^2} = \frac{\partial^2\psi(\xi)}{\partial\xi^2} \cdot \mu \cdot \delta\mu.$$

We denote partial differentiation by the round ∂, regarding ψ as a function of $\xi, p_2, p_3, p_4, \ldots$ and f as a function of $s, \rho_3, \rho_4, \rho_5, \ldots$, so that, by (7) and (27),

$$\frac{\partial\psi}{\partial\xi} = \frac{1}{\mu^2}\frac{\partial f}{\partial s} \quad \text{and} \quad \frac{\partial^2\psi}{\partial\xi^2} = \frac{1}{\mu^3}\frac{\partial^2 f}{\partial s^2}.$$

The variations denoted by δ were all taken for a *definite* value of ξ; therefore, by (7), (27), and the formulae just written,

$$\delta s = \delta\left(\frac{\xi}{\mu}\right) = -\frac{\xi}{\mu^2} \cdot \delta\mu = -s \cdot \frac{\delta\mu}{\mu},$$

$$\delta\psi = \frac{\partial^2\psi}{\partial\xi^2} \cdot \mu \cdot \delta\mu = \frac{\partial^2 f}{\partial s^2} \cdot \frac{\delta\mu}{\mu^2},$$

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* This assumption does not in any way affect the generality of the results obtained, because p_k' enters into $\psi(\xi)$ only through the variations of the unaccented p's.

$$\delta f = \delta(\mu\,\psi) = \mu \cdot \delta\psi + \psi \cdot \delta\mu = \left[\frac{\partial^2 f}{\partial s^2} + f\right] \frac{\delta\mu}{\mu},$$

$$\delta p_k = k\,(k-1) \cdot \rho_{k-2}\,\mu^{k-1} \cdot \delta\mu,$$

$$\delta\rho_k = \delta\left(\frac{p_k}{\mu^k}\right) = \frac{\delta p_k}{\mu^k} - k\,\frac{p_k}{\mu^{k+1}} \cdot \delta\mu = [k\,(k-1) \cdot \rho_{k-2} - k \cdot \rho_k]\,\frac{\delta\mu}{\mu};$$

therefore,

$$\begin{aligned}\delta f &= \frac{\partial f}{\partial s}\,\delta s + \frac{\partial f}{\partial \rho_3}\,\delta\rho_3 + \frac{\partial f}{\partial \rho_4}\,\delta\rho_4 + \frac{\partial f}{\partial \rho_5}\,\delta\rho_5 + \dots \\ &= \frac{\partial f}{\partial s}\,\delta s + \underset{3}{\sum}{}_k\,\frac{\partial f}{\partial \rho_k} \cdot \delta\rho_k \\ &= \left\{-\,s\,\frac{\partial f}{\partial s} + \underset{3}{\sum}{}_k\,[k\,(k-1)\,\rho_{k-2} - k\rho_k]\,\frac{\partial f}{\partial \rho_k}\right\}\frac{\delta\mu}{\mu}\,.\end{aligned}$$

Equating the two expressions for δf thus found, omitting the common factor $\frac{\delta\mu}{\mu}$, we obtain

$$\frac{\partial^2 f}{\partial s^2} + s\,\frac{\partial f}{\partial s} + f = \underset{3}{\sum}{}_k\,[k\,(k-1)\,\rho_{k-2} - k\,\rho_k]\,\frac{\partial f}{\partial \rho_k}, \tag{28}$$

a differential equation for f which may be written

$$\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial s} + s f\right) = 2 \cdot Tf, \tag{29}$$

where

$$T = \tfrac{1}{2}\underset{3}{\sum}{}_k\,[k\,(k-1)\,\rho_{k-2} - k\rho_k]\,\frac{\partial}{\partial \rho_k} \tag{30}$$

is a differential operator that *acts upon* f *only through the* ρ*'s with suffixes as great as* 3; the first part of that term of T for which $k = 3$ vanishes because $\rho_1 = 0$. We have omitted the upper limit to the sum in T because that depends upon the number of ρ's involved in f, a number as yet unknown.

VII. The Operator T. The P's and S's.

The operator T is so important for the theory that it deserves particular study. By a *constant* we shall henceforth mean a quantity that is independent of the ρ's with suffixes as great as 3 as well as of s. If u and v are any two operands and C is any constant, we have

$$(31) \qquad \begin{cases} T\,(u+v) = Tu + Tv, \\ T \cdot Cu = C \cdot Tu, \\ T \cdot uv = u \cdot Tv + v \cdot Tu. \end{cases}$$

If u is a function of the ρ's with suffixes not greater than m, Tu is also a function of the ρ's with suffixes not greater than m; if u is a polynomial in the ρ's, Tu is a polynomial in the ρ's of the same degree as u; in particular, if u is a linear function of the ρ's with suffixes not greater than m, Tu is a linear function of the ρ's with suffixes not greater than m. If $3 \overset{=}{<} m$,

$$(32) \qquad T\rho_m = \tfrac{1}{2}\, m\,(m-1) \cdot \rho_{m-2} - \tfrac{1}{2}\, m \cdot \rho_m.$$

Therefore, if u is a linear function of ρ_m and the ρ's with suffixes less than m, where $3 \overset{=}{<} m$, $T^n u$ is a linear function of ρ_m and the ρ's with suffixes less than m in which the coefficient of ρ_m is the coefficient of ρ_m in u multiplied by $\left(-\frac{m}{2}\right)^n$, where n is any positive integer or 0; more generally, if $G\,(T)$ denotes any polynomial in T (linear function of powers of T with constant coefficients), $G\,(T) \cdot u$ is a linear function of the ρ's with suffixes not greater than m in which the coefficient of ρ_m is the coefficient of ρ_m in u multiplied by $G\left(-\frac{m}{2}\right)$; in particular, $G\,(T) \cdot u$ will not involve ρ_m if $G\,(T)$ contains $T + \frac{m}{2}$ as a factor, and only then. Formula (32) holds also for $m = 0$, 1, and 2, giving

$$T\rho_0 = 0 \cdot \rho_0 = 0,\ T\rho_1 = -\tfrac{1}{2}\,\rho_1 = 0,\ T\rho_2 = \rho_0 - \rho_2 = 0,$$

because $\rho_0 = \rho_2 = 1$ and $\rho_1 = 0$, by (7); therefore,

$$(T + \tfrac{0}{2})\,\rho_0 = 0,\ (T + \tfrac{1}{2})\,\rho_1 = 0,\ (T + \tfrac{2}{2})\,\rho_2 = \rho_2 = 1\,;$$

so that $G(T) \cdot \rho_m$ does not involve ρ_m if $G\,(T)$ contains the factor $T + \frac{m}{2}$, where m is 0 or any positive integer excepting 2, and $G\,(T) \cdot \rho_2$ does not involve ρ_2 if $G\,(T)$ contains T as a factor. Of course, polynomials in T are commutative with each other, so that it is of no consequence what position in a product $G\,(T)$ the annihilating factor occupies.

It is convenient to employ the usual symbol $y^{(n)}$ to denote the product of n factors diminishing by successive units from y, whatever meaning y may have (it may be a symbol of quantity or of operation), so that

$$(33) \qquad y^{(n)} = y\,(y-1)\,(y-2)\,(y-3)\,\ldots\,(y-n+1)$$

if n is any positive integer; in consequence of this definition,

$$(34)\qquad \begin{cases} y^{(1)} = y, \\ y^{n+1} = y^{(n)} \cdot (y - n), \\ y^{(n)} = \dfrac{y^{(n+1)}}{y - n}, \end{cases}$$

for $1 \leqq n$; we take the last formula as the basis of the definition of $y^{(n)}$ for $n = 0, -1, -2, -3, \ldots$, successively, and thus obtain

$$(35)\qquad y^{(o)} = 1, \quad y^{(-n)} = \frac{1}{(y+1)(y+2)(y+3)\ldots(y+n)}$$

for any positive integer n (that is, for any negative integer $-n$). With this notation we have

$$(36)\qquad \begin{cases} y^{(n)} = y^{(i)} \cdot (y - i)^{(n-i)} = (-1)^n \cdot (-y + n - 1)^{(n)}, \\ (y + n - 1)^{(n)} = (-1)^n \cdot (-y)^{(n)}, \\ n^{(n)} = n\,!, \quad n^{(i)} = \dfrac{n\,!}{(n-i)\,!}, \\ \dbinom{n}{i} = \dfrac{n^{(i)}}{i\,!} = \dbinom{n-1}{i} + \dbinom{n-1}{i-1}, \end{cases}$$

where each of the letters n and i represents any positive integer or 0; also,

$$(37)\qquad \begin{cases} (2\,i)\,! = 2^{2i} \cdot i\,!\,(i - \frac{1}{2})^{(i)} \\ (2\,i + 1)\,! = 2^{2i+1} \cdot i\,!\,(i + \frac{1}{2})^{(i+1)}, \end{cases}$$

because, evidently, $2^i\, i\,!$ is the product of the even numbers from 2 to $2\,i$, $2^i\,(i - \frac{1}{2})^{(i)}$ is the product of the odd numbers from 1 to $2\,i - 1$, and $2^{i+1}\,(i + \frac{1}{2})^{(i+1)}$ is the product of the odd numbers from 1 to $2\,i + 1$. Furthermore, we have

$$(38)\qquad \begin{aligned} (y-1)^{(n)} &= (y-1)^{(n-1)} \cdot (y - n) = y^{(n)} - n \cdot (y-1)^{(n-1)} \\ &= y^{(n)} - n \cdot y^{(n-1)} + n^{(2)} \cdot y^{(n-2)} - n^{(3)} \cdot y^{(n-3)} + \ldots \\ &= \sum_{0}^{n}{}_{k} (-1)^k \cdot n^{(k)} \cdot y^{(n-k)}, \end{aligned}$$

where n is any positive integer or 0.

From (32) it is evident that any polynomial in T changes ρ with any even suffix into a linear function of the ρ's with even suffixes and ρ with any odd suffix into a linear function of the ρ's with odd suffixes. Therefore, if u is a linear function of ρ_{2m} and the ρ's with even suffixes less than $2\,m$, where $2 \leqq m$, $(T + m)\,u$ does not involve ρ_{2m}, $(T + m)^{(2)}\,u$ involves neither ρ_{2m} nor ρ_{2m-2}, $(T + m)^{(3)}\,u$ involves neither ρ_{2m}, ρ_{2m-2}, nor ρ_{2m-4}, and so on, and $(T + m)^{(m-1)}\,u$ is a constant (or linear function of ρ_0 and ρ_2) ; therefore, by (36)

$$(39)\qquad \left\{\begin{aligned} (T + m)^{(m+1)}\,u &= (T + m)^{(m-1)}\,(T + 1)^{(2)}\,u \\ &= (T + 1)\,T \cdot (T + m)^{(m-1)}\,u = 0\,; \end{aligned}\right.$$

that is, $(T + m)^{(m+1)}$ annihilates every linear function of the ρ's with even suffixes not greater than $2\,m$, where m is any positive integer or 0 (this is true for $m = 0$ or 1 because the operator contains T as a factor). But $(T + m)^{(m+1)}$ annihilates no linear function that involves a ρ with an even suffix greater than $2\,m$ or an odd suffix as great as 3 ; namely, if ρ_k is the ρ with greatest even suffix or greatest odd suffix (other than 1) that is involved in the linear function u (with a coefficient different from 0), the coefficient of ρ_k in $(T + m)^{(m+1)}\,u$ is the coefficient of ρ_k in u multiplied by $\left(m - \frac{k}{2}\right)^{(m+1)}$, whose factors are all negative if k is an even number greater than $2\,m$ and all odd multiples of $\frac{1}{2}$ if k is odd.

Similarly, if u is a linear function of ρ_{2m+1} and the ρ's with odd suffixes less than $2\,m + 1$, where $0 \leqq m$,

$$(40)\qquad (T + m + \tfrac{1}{2})^{(m+1)}\,u = 0,$$

so that $(T + m + \frac{1}{2})^{(m+1)}$ annihilates any linear function of the ρ's with odd suffixes not greater than $2\,m + 1$; but $(T + m + \frac{1}{2})^{(m+1)}$ annihilates no linear function that involves a ρ with an odd suffix greater than $2\,m + 1$ or an even suffix.

Applying (39) and (40) to ρ_{2m} and ρ_{2m+1}, respectively, we have

$$(T + m)^{(m+1)}\,\rho_{2m} = 0, \quad (T + m + \tfrac{1}{2})^{(m+1)}\,\rho_{2m+1} = 0,$$

so that, by (34),

$$T\,(T + m - 1)^{(m)}\,\rho_{2m} = -\,m \cdot (T + m - 1)^{(m)}\,\rho_{2m},$$

$$T\,(T + m - \tfrac{1}{2})^{(m)}\,\rho_{2m+1} = -\,(m + \tfrac{1}{2}) \cdot (T + m - \tfrac{1}{2})^{(m)}\,\rho_{2m+1},$$

or

$$(41)\qquad T\,Q_{2m} = -\,m \cdot Q_{2m}, \quad T\,Q_{2m+1} = -\,(m + \tfrac{1}{2}) \cdot Q_{2m+1},$$

if we write

$$(42) \qquad (T+m-1)^{(m)}\rho_{2m} = Q_{2m}, \qquad (T+m-\tfrac{1}{2})^{(m)}\rho_{2m+1} = Q_{2m+1};$$

and, evidently, formulae (41) hold for any constant multiples of Q_{2m} and Q_{2m+1}, respectively.

Now (32) shows that

$$(T+m)\cdot\rho_{2m} = 2\,m\,(m-\tfrac{1}{2})\cdot\rho_{2m-2}$$

and, by continued application of the same formula,

$$(43) \qquad (T+m)^{(k)}\rho_{2m} = 2^k\,m^{(k)}\,(m-\tfrac{1}{2})^{(k)}\cdot\rho_{2m-2k}$$

for $0 \leqq k \leqq m-1$; in particular,

$$(T+m)^{(m-1)}\rho_{2m} = 2^{m-1}\,m^{(m-1)}\,(m-\tfrac{1}{2})^{(m-1)}\,\rho_2$$

and

$$(44) \qquad (T+m)^{(m)}\rho_{2m} = 2^m\cdot m!\,(m-\tfrac{1}{2})^{(m)}\rho_0,$$

because, by (34) and (36),

$$m! = m^{(m)} = m^{(m-1)}\cdot 1,$$
$$(m-\tfrac{1}{2})^{(m)} = (m-\tfrac{1}{2})^{(m-1)}\cdot\tfrac{1}{2},$$
$$(T+1)\,\rho_2 = \rho_2 = \rho_0;$$

so that (43) holds also for $k=m$. Then, by (42), (38), (43), and (36),

$$(45) \qquad \begin{aligned} Q_{2m} &= \sum_{0}^{m}{}_{i}\,(-1)^i\,m^{(i)}\,(T+m)^{(m-i)}\,\rho_{2m} \\ &= 2^m\,(m-\tfrac{1}{2})^{(m)}\cdot m!\sum_{0}^{m}{}_{i}\,(-1)^i\binom{m}{i}\frac{\rho_{2i}}{2^i\,(i-\frac{1}{2})^{(i)}} \end{aligned}$$

for $0 \leqq m$.

Similarly, we have, from (32),

$$(T+m+\tfrac{1}{2})\cdot\rho_{2m+1} = 2\,m\,(m+\tfrac{1}{2})\cdot\rho_{2m-1}$$

and, by continued application,

$$(46) \qquad (T+m+\tfrac{1}{2})^{(k)}\cdot\rho_{2m+1} = 2^k\,m^{(k)}\,(m+\tfrac{1}{2})^{(k)}\cdot\rho_{2m-2k+1}$$

for $0 \leqq k \leqq m$. Therefore, by (42), (38), (46), and (36),

$$(47) \qquad \begin{aligned} Q_{2m+1} &= \sum_{0}^{m}{}_{i}\,(-1)^i\,m^{(i)}\,(T+m+\tfrac{1}{2})^{(m-i)}\,\rho_{2m+1} \\ &= 2^{m+1}\,(m+\tfrac{1}{2})^{(m+1)}\cdot m!\sum_{0}^{m}{}_{i}\,(-1)^i\binom{m}{i}\frac{\rho_{2i+1}}{2^{i+1}\,(i+\frac{1}{2})^{(i+1)}} \end{aligned}$$

for $0 \leqq m$.

If we write

$$(48)\qquad \begin{cases} P_{2m} = \sum_{0}^{m}{}_i (-1)^i \binom{m}{i} \dfrac{\rho_{2i}}{2^i\,(i-\frac{1}{2})^{(i)}} \qquad \text{and} \\[2ex] P_{2m+1} = \sum_{0}^{m}{}_i (-i)^i \binom{m}{i} \dfrac{\rho_{2i+1}}{2^{i+1}\,(i+\frac{1}{2})^{(i+1)}}, \end{cases}$$

we have, by (45) and (47),

$$(49)\qquad \begin{cases} Q_{2m} = 2^m\,(m-\tfrac{1}{2})^{(m)}\,m\,!\,P_{2m} \qquad \text{and} \\ Q_{2m+1} = 2^{m+1}\,(m+\tfrac{1}{2})^{(m+1)}\,m\,!\,P_{2m+1}, \end{cases}$$

and, because P_{2m} and P_{2m+1} are constant multiples of Q_{2m} and Q_{2m+1}, by (41),

$$(50)\qquad TP_{2m} = -\,m \cdot P_{2m}, \qquad TP_{2m+1} = -\,(m+\tfrac{1}{2}) \cdot P_{2m+1},$$

for all values of m. For the P's with the smaller suffixes we have, from (48) and (7),

$$(51)\qquad \begin{cases} P_0 = 1, \quad P_1 = P_2 = 0, \quad P_3 = -\tfrac{1}{3}\rho_3, \quad P_4 = -\,1 + \tfrac{1}{3}\rho_4, \\ P_5 = -\,\tfrac{2}{3}\rho_3 + \tfrac{1}{15}\rho_5, \quad P_6 = -\,2 + \rho_4 - \tfrac{1}{15}\rho_6 \quad \text{etc.}; \end{cases}$$

so that equations (50) give

$$(52)\qquad TP_0 = 0 \cdot P_0 = 0, \quad TP_1 = -\tfrac{1}{2}P_1 = 0, \quad TP_2 = -\,P_2 = 0.$$

Because P_{2m} is a linear function of the ρ's with even suffixes up to $2\,m$ in which the coefficient of ρ_{2m} is different from 0, and P_{2m+1} is a linear function of the ρ's with odd suffixes up to $2\,m+1$ in which the coefficient of ρ_{2m+1} is different from 0, ρ_{2m} can be expressed as a linear function of the P's with even suffixes up to $2\,m$ and ρ_{2m+1} as a linear function of the P's with odd suffixes up to $2\,m+1$. In fact, if we multiply P_{2m} or P_{2m+1} by $(-1)^m \binom{k}{m}$ and sum for all values of m from 0 to k, we obtain

$$\sum_{0}^{k}{}_m (-1)^m \binom{k}{m} P_{2m} = \sum_{0}^{k}{}_i (-1)^i \frac{\rho_{2i}}{2^i\,(i-\frac{1}{2})^{(i)}} \sum_{0}^{k}{}_m (-1)^m \binom{k}{m}\binom{m}{i}$$

$$= \frac{\rho_{2k}}{2^k\,(k-\frac{1}{2})^{(k)}},$$

$$\sum_{0}^{k}{}_m (-1)^m \binom{k}{m} P_{2m+1} = \sum_{0}^{k}{}_i (-1)^i \frac{\rho_{2i+1}}{2^{i+1}\,(i+\frac{1}{2})^{(i+1)}} \sum_{i}^{k}{}_m (-1)^m \binom{k}{m}\binom{m}{i}$$

$$= \frac{\rho_{2k+1}}{2^{k+1}\,(k+\frac{1}{2})^{(k+1)}},$$

because

$$\sum_{i}^{k}{}_{m}(-1)^{m}\binom{k}{m}\binom{m}{i}=\binom{k}{i}\sum_{i}^{k}{}_{m}(-1)^{m}\binom{k-i}{m-i}$$

$$=(-1)^{i}\binom{k}{i}\sum_{0}^{k-i}{}_{m}(-1)^{m}\binom{k-i}{m}$$

$$=0 \text{ if } i<k \text{ and}$$

$$=(-1)^{k} \text{ if } i=k;$$

therefore

$$(53)\quad \begin{cases} \rho_{2k}=2^{k}(k-\tfrac{1}{2})^{(k)}\sum_{0}^{k}{}_{m}(-1)^{m}\binom{k}{m}P_{2m} & \text{for } 0 \leqq k, \\ \rho_{2k+1}=2^{k+1}(k+\tfrac{1}{2})^{(k+1)}\sum_{0}^{k}{}_{m}(-1)^{m}\binom{k}{m}P_{2m+1} & \text{for } 0 \leqq k. \end{cases}$$

As a consequence of (50) we have

$$(54)\qquad G(T)\cdot P_{m}=G\left(-\frac{m}{2}\right)\cdot P_{m},$$

if $G(T)$ is any polynomial in T. The ease with which the result of applying any polynomial in T to the P's can be obtained by means of this formula makes the introduction of the P's in place of the ρ's very advantageous. We shall henceforth assume that all functions of the ρ's have been expressed as functions of the P's by (53), so that the degree of any polynomial in the P's with suffixes as great as 3 is the same as that of the corresponding polynomial in the ρ's. Equations (50) show also that the operator T, defined by (30) in terms of the ρ's, can be expressed in terms of the P's thus:

$$(55)\qquad T=-\tfrac{1}{2}\sum_{3}{}_{k}\,k\,P_{k}\frac{\partial}{\partial P_{k}}.$$

We assign to each P a *weight* equal to its suffix (leaving out of account P_1 and P_2, which are 0 and, therefore, nowhere occur) and define the weight of any product of P's to be the sum of the weights of its factors; any constant may be regarded as a multiple of $P_0 = 1$, and, therefore, as of the weight 0. The weight of any such product as $P_3^{n_3}P_4^{n_4}P_5^{n_5}\ldots$ is

$$(56)\qquad 3n_3+4n_4+5n_5+\ldots=w,$$

say, so that, by (55) and the last formula of (31),

$$(57)\qquad T\cdot P_3^{n_3}P_4^{n_4}P_5^{n_5}\ldots=-\frac{w}{2}\cdot P_3^{n_3}P_4^{n_4}P_5^{n_5}\ldots$$

and, evidently

$$(58)\qquad G(T)\cdot P_3^{n_3}P_4^{n_4}P_5^{n_5}\ldots = G\left(-\frac{w}{2}\right)\cdot P_3^{n_3}P_4^{n_4}P_5^{n_5}\ldots$$

where $G(T)$ is any polynomial in T. More generally, if Π is any *isobaric* polynomial of weight w in the P's, that is, any polynomial whose terms are *all* of one weight w,

$$(59)\qquad G(T)\cdot\Pi = G\left(-\frac{w}{2}\right)\cdot\Pi.$$

Therefore, if the coefficients of Π are not all 0, $G(T)\cdot\Pi$ is or is not 0 according as $G(T)$ does or does not contain $T+\frac{w}{2}$ as a factor. In particular, $(T+m)^{(m+1)}$ annihilates every polynomial in the P's whose terms are all of even weights not greater than $2m$ and annihilates no polynomial of other description, while $(T+m+\frac{1}{2})^{(m+1)}$ annihilates every polynomial whose terms are all of odd weights not greater than $2m+1$ and no others. By means of formula (59) any polynomial in T can be applied to any polynomial in the P's; namely, any polynomial in the P's can be expressed as the sum of isobaric polynomials of different weights and the polynomial in T can be applied to each isobaric part by (59).

The P's can be determined from the function $f(s)$; namely, from (7), (13), and (27) we have

$$(60)\qquad \rho_k = \frac{p_k}{\mu^k} = \int_{\underline{\xi}}^{\bar{\xi}} \frac{\xi^k}{\mu^k}\psi(\xi)\cdot d\xi = \int_{\underline{s}}^{\bar{s}} s^k\cdot f(s)\cdot ds \qquad \text{for } 0 \leqq k,$$

and obtain, by the substitution of these values of the ρ's in (48),

$$(61)\qquad P_{2m} = \int_{\underline{s}}^{\bar{s}} S_{2m}\cdot f(s)\cdot ds,\qquad P_{2m+1} = \int_{\underline{s}}^{\bar{s}} S_{2m+1}\cdot f(s)\,.\,ds,$$

where

$$(62)\qquad \begin{cases} S_{2m} = \sum\limits_{0}^{m}{}_i(-1)^i\dbinom{m}{i}\dfrac{s^{2i}}{2^i\,(i-\frac{1}{2})^{(i)}}, \\[2ex] S_{2m+1} = \sum\limits_{0}^{m}{}_i(-i)^i\dbinom{m}{i}\dfrac{s^{2i+1}}{2^{i+1}\,(i+\frac{1}{2})^{(i+1)}}, \end{cases}$$

for $0 \leqq m$. For the S's with the smaller suffixes we have

$$(63)\qquad \begin{cases} S_0 = 1, \quad S_1 = s, \quad S_2 = 1 - s^2, \quad S_3 = s - \frac{1}{3}s^3, \\ S_4 = 1 - 2\,s^2 + \frac{1}{3}s^4, \quad S_5 = s - \frac{2}{3}s^3 + \frac{1}{15}s^5, \\ S_6 = 1 - 3\,s^2 + s^4 - \frac{1}{15}s^6, \text{ etc.} \end{cases}$$

VIII. THE FORM OF $f(s)$. THE R'S.

To simplify the integration of equation (29), we put

$$f(s) = \theta\,(s) \cdot u,$$

where u is an integral of the equation

$$\frac{\partial u}{\partial s} + s \cdot u = 0,$$

say

$$u = e^{-\frac{s^2}{2}},$$

where e represents the base of natural logarithms, so that

$$(64)\qquad f(s) = \theta\,(s) \cdot e^{-\frac{s^2}{2}}.$$

Writing θ as an abbreviation for $\theta\,(s)$, we have

$$\frac{\partial f}{\partial s} = \left[\frac{\partial \theta}{\partial s} - s \cdot \theta\right] e^{-\frac{s^2}{2}},$$

$$\frac{\partial^2 f}{\partial s^2} = \left[\frac{\partial^2 \theta}{\partial s^2} - 2\,s \cdot \frac{\partial \theta}{\partial s} - (1 - s^2)\,\theta\right] e^{-\frac{s^2}{2}},$$

so that (29) becomes

$$\left[\frac{\partial^2 \theta}{\partial s^2} - s \cdot \frac{\partial \theta}{\partial s}\right] e^{-\frac{s^2}{2}} = 2 \cdot T\theta \cdot e^{-\frac{s^2}{2}},$$

because $e^{-\frac{s^2}{2}}$ is independent of the ρ's (and P's) ; therefore,

$$(65)\qquad \frac{\partial^2 \theta}{\partial s^2} - s \cdot \frac{\partial \theta}{\partial s} = 2 \cdot T\theta,$$

which is a differential equation for $\theta\,(s)$.

The assumption II that $\phi\,(x)$ is developable by Taylor's theorem throughout the range of possible values of x implies that $\psi\,(\xi)$, $f\,(s)$, and $\theta\,(s)$, are similarly developable; we may therefore put

$$(66)\qquad \theta\,(s) = \sum_{0}{}_{i} \frac{A_i}{i\,!}\, s^i,$$

where A_i (for each value of i) is a function of the P's alone. This development implies that we shall approximate more and more nearly to the true form of $\theta(s)$, even if its complete expression is an infinite series, the more terms of (66) we take into account. Taking any finite number of terms, say for values of i up to κ, substituting in (64) and the resulting expression for $f(s)$ in (61), we obtain

$$(67) \qquad P_m = \sum_{0}^{\kappa}{}_i \frac{A_i}{i!} \int_{\underline{s}}^{\bar{s}} s^i R_m \cdot ds,$$

where

$$(68) \qquad R_m = S_m \cdot e^{-\frac{s^2}{2}} \text{ for } 0 \overset{=}{<} m.$$

We see from (67) that, to the degree of approximation that corresponds to $\kappa + 1$ terms of (66), the P's are linear functions of the $\kappa + 1$ A's, whose coefficients are definite functions of the limits $\underline{s}$ and $\bar{s}$ alone, because the S's and, therefore, the R's are independent of the P's. Therefore, not more than $\kappa + 1$ of the P's (including P_0, P_1, and P_2) are *linearly* independent; the solution of such equations (67) as correspond to linearly independent P's gives the $\kappa + 1$ A's as linear functions of these P's, whose coefficients are definite functions of $\underline{s}$ and $\bar{s}$ derived from the integrals involved in (67); these expressions of the A's will involve a certain number of arbitrary constants if the number of linearly independent P's is less than $\kappa + 1$, but that is of no consequence. If the P's are subjected to infinitesimal variations, — say δP_m is the variation of P_m, — the A's, $f(s)$, and $\theta(s)$ will undergo variations, which may be represented by δA_i (for A_i), $\delta f(s)$, and $\delta\theta(s)$, respectively. Then, by (66) and (64),

$$\delta\theta(s) = \sum{}_i \frac{\delta A_i}{i!} s^i, \qquad \delta f(s) = \delta\theta(s) \cdot e^{-\frac{s^2}{2}},$$

and, by (61) and (68), because $f(s) = 0$ for each of the limits $\underline{s}$ and $\bar{s}$,

$$\delta P_m = \int_{\underline{s}}^{\bar{s}} S_m \cdot \delta f(s) \cdot ds = \int_{\underline{s}}^{\bar{s}} R_m \cdot \delta\theta(s) \cdot ds = \sum_{0}{}_i \frac{\delta A_i}{i!} \int_{\underline{s}}^{\bar{s}} s^i R_m \cdot ds;$$

that is, the relations between the δP's and the δA's are the same as if the coefficients of the A's in (67) were constants (whether $\underline{s}$ and $\bar{s}$ are functions of the P's or not); therefore, in determining the A's as functions of the P's that satisfy the differential equation (65) we may regard them as linear functions of the P's with *constant* coefficients. The necessary and sufficient conditions for these linear functions are those implied in (65) together with

(69) $$A_i = 0 \text{ for } \kappa < i$$

(to the degree of approximation to which we restrict our calculations).

Substituting the expression (66) for $\theta(s)$ in (65) we obtain

$$\sum_{2}^{\kappa}{}_{i} \frac{A_i}{(i-2)!} s^{i-2} - \sum_{1}^{\kappa}{}_{i} \frac{A_i}{(i-1)!} s^{i} = 2 \sum_{0}^{\kappa}{}_{i} \frac{TA_i}{i!} s^{i},$$

that is

$$\sum_{0}^{\kappa}{}_{i} \left[A_{i+2} - i A_i - 2 \cdot TA_i \right] \frac{s^i}{i!} = 0$$

for every possible value of s, where the first term of the multiplier of $\frac{s^i}{i!}$ is to be omitted for $i = \kappa$ and $i = \kappa - 1$ and the second term goes out of itself for $i = 0$. Therefore, the necessary and sufficient conditions for the satisfaction of (65) by (66) are

$$A_{i+2} = 2\left(T + \frac{i}{2}\right) \cdot A_i \text{ for } 0 \leqq i \leqq \kappa - 2$$

and

$$\left(T + \frac{i}{2}\right) \cdot A_i = 0 \text{ for } i = \kappa \text{ or } \kappa - 1;$$

that is

(70) $$A_i = 2\left(T + \frac{i-2}{2}\right) \cdot A_{i-2} \text{ for } 2 \leqq i \leqq \kappa$$

and

(71) $$\left(T + \frac{i}{2}\right) \cdot A_i = 0$$

for A_i with the greatest even suffix and A_i with the greatest odd suffix that occur in (66). On replacing i in (70) by $2i$ and $2i + 1$, respectively, we have, for the A's with even and odd suffixes separately,

$$A_{2i} = 2(T + i - 1) \cdot A_{2i-2} \quad \text{for} \quad 1 \leqq i \leqq \frac{\kappa}{2},$$

$$A_{2i+1} = 2(T + i - \tfrac{1}{2}) \cdot A_{2i-1} \quad \text{for} \quad 1 \leqq i \leqq \frac{\kappa - 1}{2},$$

and, by repeated application of these formulae,

(72) $$A_{2i} = 2^i (T + i - 1)^{(i)} \cdot A_0 \quad \text{for} \quad 1 \leqq i \leqq \frac{\kappa}{2}$$

and

(73) $$A_{2i+1} = 2^i (T + i - \tfrac{1}{2})^{(i)} \cdot A_1 \quad \text{for} \quad 1 \leqq i \leqq \frac{\kappa - 1}{2};$$

that is, the A's with even suffixes are all derived from A_0 and all the A's with odd suffixes from A_1 by the application of definite polynomials in T. We may, then, determine the forms of the A's with even suffixes and of those with odd suffixes separately; that is, we may suppose our calculations restricted to a degree of approximation that corresponds to a development of $\theta(s)$ in which occur even powers of s up to $s^{2\alpha}$ and odd powers up to $s^{2\beta+1}$, without assuming any particular relation between α and β. Then (72) holds for $1 \leqq i \leqq \alpha$, (73) holds for $1 \leqq i \leqq \beta$, while (71) is expressed by

$$(T+\alpha)\cdot A_{2\alpha}=0,\ (T+\beta+\tfrac{1}{2})\cdot A_{2\beta+1}=0,$$

that is, when $A_{2\alpha}$ and $A_{2\beta+1}$ have been determined by (72) and (73), respectively,

$$(74)\qquad (T+\alpha)^{(\alpha+1)}\cdot A_0=0,\ (T+\beta+\tfrac{1}{2})^{(\beta+1)}\cdot A_1=0.$$

The only conditions imposed upon A_0 and A_1 so far are, then, that they shall be linear functions of the P's with constant coefficients that satisfy the equations (74). The only *linear* functions of the P's that are of a given weight w are constant multiples of P_w, while, by (54),

$$(T+\alpha)^{(\alpha+1)}\cdot P_w=\left(\alpha-\frac{w}{2}\right)^{(\alpha+1)}\cdot P_w,$$

$$(T+\beta+\tfrac{1}{2})^{(\beta+1)}\cdot P_w=\left(\beta-\frac{w-1}{2}\right)^{(\beta+1)}\cdot P_w;$$

therefore, in order that A_0 (as a linear function of the P's) shall satisfy the first equation (74) it is necessary and sufficient that the suffix w of any P involved in it shall satisfy the equation

$$\left(\alpha-\frac{w}{2}\right)^{(\alpha+1)}=0,$$

that is that w shall be an *even* integer not greater than 2α; and, in order that A_1 (as a linear function of the P's) shall satisfy the second equation (74) it is necessary and sufficient that the suffix w of any P involved in it shall satisfy the equation

$$\left(\beta-\frac{w-1}{2}\right)^{(\beta+1)}=0,$$

that is that w shall be an *odd* integer not greater than $2\beta+1$. We may, then, write

$$(75)\qquad A_0=\sum_{0}^{\alpha}{}_k\, C_{2k}P_{2k},\ A_1=\sum_{0}^{\beta}{}_k\, C_{2k+1}P_{2k+1},$$

where the C's are constants.

On applying (72) and (73) to the first and second of equations (75), respectively, we obtain, by (54) and (36),

$$(76)\begin{cases} A_{2i} = \sum\limits_{0}^{\alpha}{}_{k}\, C_{2k} \,.\, P_{2k} \cdot 2^i\,(i-k-1)^{(i)} = \sum\limits_{0}^{\alpha}{}_{k}\, C_{2k} P_{2k} \cdot (-1)^i\, 2^i k^{(i)}, \\ A_{2i+1} = \sum\limits_{0}^{\beta}{}_{k}\, C_{2k+1} P_{2k+1} \cdot 2^i\,(i-k-1)^{(i)} = \sum\limits_{0}^{\beta}{}_{k}\, C_{2k+1} P_{2k+1} \cdot (-1)^i\, 2^i k^{(i)}. \end{cases}$$

Substituting these values of A_{2i} and A_{2i+1} in (66) we have

$$\theta\,(s) = \sum\limits_{0}^{\alpha}{}_{k}\, C_{2k}\, P_{2k} \cdot \sum\limits_{0}^{k}{}_{i}\,(-1)^i\, \frac{2^i\, k^{(i)}}{(2\,i)\,!}\, s^{2i}$$

$$+ \sum\limits_{0}^{\beta}{}_{k}\, C_{2k+1}\, P_{2k+1} \cdot \sum\limits_{0}^{k}{}_{i}\,(-1)^i\, \frac{2^i\, k^{(i)}}{(2\,i+1)\,!}\, s^{2i+1}$$

or, replacing $(2\,i)\,!$ and $(2\,i + 1)\,!$ by their values in (37), observing the last line of (36), and employing the notation of (62),

$$(77) \qquad \theta\,(s) = \sum\limits_{0}^{\alpha}{}_{k}\, C_{2k}\, P_{2k}\, S_{2k} + \sum\limits_{0}^{\beta}{}_{k}\, C_{2k+1}\, P_{2k+1}\, S_{2k+1}\,;$$

again, by (64) and (68),

$$(78) \qquad f\,(s) = \sum\limits_{0}^{\alpha}{}_{k}\, C_{2k}\, P_{2k}\, R_{2k} + \sum\limits_{0}^{\beta}{}_{k}\, C_{2k+1}\, P_{2k+1}\, R_{2k+1}\,.$$

Substituting this expression of $f\,(s)$ in (61) we obtain

$$(79)\begin{cases} P_{2m} = \sum\limits_{0}^{\alpha}{}_{k}\, C_{2k} P_{2k} \cdot \int_{\underline{s}}^{\bar{s}} S_{2m} R_{2k} \cdot ds + \sum\limits_{0}^{\beta}{}_{k}\, C_{2k+1} P_{2k+1} \cdot \int_{\underline{s}}^{\bar{s}} S_{2m} R_{2k+1} \cdot ds, \\ P_{2m+1} = \sum\limits_{0}^{\alpha}{}_{k}\, C_{2k} P_{2k} \cdot \int_{\underline{s}}^{\bar{s}} S_{2m+1} R_{2k} \cdot ds + \sum\limits_{0}^{\beta}{}_{k}\, C_{2k+1} P_{2k+1} \cdot \int_{\underline{s}}^{\bar{s}} S_{2m+1} R_{2k+1} \cdot ds. \end{cases}$$

IX. THE COEFFICIENTS OF $f(s)$. CONDITIONS FOR THE LIMITS.

In all that contributed to the determination of the form of $f\,(s)$ as given by (78) no assumption has been made with regard to the *values* of the P's with even suffixes not greater than $2\,\alpha$ and the P's with odd suffixes not greater than $2\,\beta + 1$ [excepting that, by (51), $P_0 = 1$,

$P_1 = P_2 = 0$], — in fact, they have been assumed to be entirely arbitrary; therefore, the first of equations (79) for $0 \leqq m \leqq \alpha$ and the second of these equations for $0 \leqq m \leqq \beta$ must be satisfied independently of the values of the P's involved; that is, for any one of the values of m in question, the coefficient of any P in the right member of the corresponding equation must be 0 excepting the coefficient of the P that occurs in the left member, which must be 1. Therefore,

$$(80)\quad \begin{cases} C_{2m} \int_{\underline{s}}^{\overline{s}} S_{2m} R_{2m} \cdot ds = 1 \quad \text{for} \quad 0 \leqq m \leqq \alpha, \ \text{excepting} \ m = 1; \\ C_{2m+1} \int_{\underline{s}}^{\overline{s}} S_{2m+1} R_{2m+1} \cdot ds = 1 \quad \text{for} \quad 1 \leqq m \leqq \beta; \end{cases}$$

$$(81)\quad \begin{cases} \int_{\underline{s}}^{\overline{s}} S_{2m} R_{2k} \cdot ds = 0 \quad \text{for} \quad 0 \leqq m \leqq \alpha, 0 \leqq k \leqq \alpha, k \lessgtr m; \\ \int_{\underline{s}}^{\overline{s}} S_{2m} R_{2k+1} \cdot ds = 0 \quad \text{for} \quad 0 \leqq m \leqq \alpha, 0 \leqq k \leqq \beta; \\ \int_{\underline{s}}^{\overline{s}} S_{2m+1} R_{2k} \cdot ds = 0 \quad \text{for} \quad 0 \leqq m \leqq \beta, 0 \leqq k \leqq \alpha; \\ \int_{\underline{s}}^{\overline{s}} S_{2m+1} R_{2k+1} \cdot ds = 0 \quad \text{for} \quad 0 \leqq m \leqq \beta, 0 \leqq k \leqq \beta, k \lessgtr m. \end{cases}$$

From the first, second, third, and fourth sets of equations (81) we ought to exclude cases for which $k = 1$, 0, 1, and 0, respectively, because $P_1 = P_2 = 0$ and the terms corresponding to these cases do not occur in (79); but, by (68),

$$(82)\qquad \int_{\underline{s}}^{\overline{s}} S_m R_k \cdot ds = \int_{\underline{s}}^{\overline{s}} S_m S \cdot e^{-\frac{s^2}{2}} \cdot ds = \int_{\underline{s}}^{\overline{s}} S_k R_m \cdot ds,$$

so that the cases that are to be excluded from any one set of equations (81) will occur (with an interchange of m and k) in other sets, excepting only the case of the integral

$$\int_{\underline{s}}^{\bar{s}} S_2 R_1 \cdot ds = \int_{\underline{s}}^{\bar{s}} S_1 R_2 \cdot ds,$$

which does not occur as the left member of any equation. The four sets of equations (81) can be represented, by virtue of (82), as the one set

$$(81') \qquad \int_{\underline{s}}^{\bar{s}} S_m R_k \cdot ds = 0$$

for any two *different* numbers m and k from among the even numbers not greater than $2\,\alpha$ (including 0) and the odd numbers not greater than $2\,\beta + 1$, excepting $m = 1$, $k = 2$, and $m = 2$, $k = 1$. The equations (80) determine all the constant coefficients C that actually occur in (78) (C_1 and C_2 do not occur), while equations (81) express relations that must exist between the limits $\underline{s}$ and $\bar{s}$ for any observations that conform to the theory.

To determine the values of the integrals that occur in (80) and (81), we consider the functions S and R defined by (62) and (68). We have

$$\frac{\partial S_{2m}}{\partial s} = \sum_{1}^{m}{}_i (-1)^i \binom{m}{i} \frac{i\, s^{2i-1}}{2^{i-1}\,(i - \frac{1}{2})^{(i)}}$$

$$= 2\,m \sum_{1}^{m}{}_i (-1)^i \binom{m-1}{i-1} \frac{s^{2i-1}}{2^i\,(i - \frac{1}{2})^{(i)}}$$

$$= -\,2\,m \sum_{0}^{m-1}{}_i (-1)^i \binom{m-1}{i} \frac{s^{2i+1}}{2^{i+1}\,(i - \frac{1}{2})^{(i+1)}},$$

$$\frac{\partial S_{2m+1}}{\partial s} = \sum_{0}^{m}{}_i (-1)^i \binom{m}{i} \frac{s^{2i}}{2^i\,(i - \frac{1}{2})^{(i)}}\,;$$

that is,

$$(83) \qquad \frac{\partial S_{2m}}{\partial s} = -\,2\,m \cdot S_{2m-1}\,, \qquad \frac{\partial S_{2m+1}}{\partial s} = S_{2m}\,,$$

for $0 \leqq m$. We have also, by (36),

$$S_{2m} = \sum_{0}^{m-1}{}_i (-1)^i \binom{m-1}{i} \frac{s^{2i}}{2^i\,(i - \frac{1}{2})^{(i)}} + \sum_{1}^{m}{}_i (-1)^i \binom{m-1}{i-1} \frac{s^{2i}}{2^i (i - \frac{1}{2})^{(i)}}$$

$$= S_{2m-2} - \sum_{0}^{m-1}{}_i (-1)^i \binom{m-1}{i} \frac{s^{2i+2}}{2^{i+1}\,(i + \frac{1}{2})^{(i+1)}}$$

$$(84)\qquad = S_{2m-2} - s \cdot S_{2m-1} \qquad \text{for } 0 \leqq m,$$

$$s \cdot S_{2m} = \sum_{0}^{m}{}_{i} (-1)^i \binom{m}{i} \left[2\,m + 1 - 2\,(m - i)\right] \frac{s^{2i+1}}{2^{i+1}\,(i + \frac{1}{2})^{(i+1)}}$$

$$= (2\,m + 1) \cdot S_{2m+1} - 2\,m \sum_{0}^{m-1}{}_{i} (-1)^i \binom{m-1}{i} \frac{s^{2i+1}}{2^{i+1}\,(i + \frac{1}{2})^{(i+1)}}$$

$$= (2\,m + 1) \cdot S_{2m+1} - 2\,m \cdot S_{2m-1}\,,$$

that is,

$$(85)\qquad S_{2m+1} = \frac{2\,m}{2\,m + 1} \cdot S_{2m-1} + \frac{1}{2\,m + 1} \cdot s \cdot S_{2m} \quad \text{for } 0 \leqq m.$$

From (68), (84), and (85) we obtain

$$(86)\qquad \begin{cases} R_{2m} = R_{2m-2} - s \cdot R_{2m-1} & \text{for } 0 \leqq m, \\ R_{2m+1} = \dfrac{2\,m}{2\,m + 1} \cdot R_{2m-1} + \dfrac{1}{2\,m + 1} \cdot s \cdot R_{2m} & \text{for } 0 \leqq m, \end{cases}$$

while from (68), (83), (84), and (85) follow

$$\frac{\partial R_{2m}}{\partial s} = -\,(2\,m \cdot S_{2m-1} + s \cdot S_{2m})\,e^{-\frac{s^2}{2}} = -\,(2\,m + 1)\,S_{2m+1} \cdot e^{-\frac{s^2}{2}}$$

$$(87)\qquad = -\,(2\,m + 1) \cdot R_{2m+1}\,,$$

$$\frac{\partial R_{2m+1}}{\partial s} = (S_{2m} - s \cdot S_{2m+1})\,e^{-\frac{s^2}{2}} = S_{2m+2} \cdot e^{-\frac{s^2}{2}}$$

$$(88)\qquad = R_{2m+2}\,.$$

From (83), (87), and (88) follow, after simple reductions by (84)–(86),

$$\frac{\partial}{\partial s}\,\{m \cdot S_{2m-1}\,R_{2k} - k \cdot S_{2m}\,R_{2k-1}\} = (m - k) \cdot S_{2m}\,R_{2k}\,,$$

$$\frac{\partial}{\partial s}\,\{S_{2m+1}\,R_{2k} - S_{2m}\,R_{2k+1}\} = 2\,(m - k) \cdot S_{2m+1}\,R_{2k+1}\,,$$

$$\frac{\partial}{\partial s}\,\{2\,m \cdot S_{2m-1}\,R_{2k+1} + S_{2m}\,R_{2k}\} = (2\,m - 2\,k - 1) \cdot S_{2m}\,R_{2k+1}\,.$$

Therefore, letting $[\;]_{\underline{s}}^{\bar{s}}$ denote the excess of the value of the enclosed function of s for $s = \bar{s}$ over its value for $s = \underline{s}$, we have

$$(89)\qquad \int_{\underline{s}}^{\bar{s}} S_{2m}\,R_{2k} \cdot ds = \frac{m}{m - k}\,[S_{2m-1}\,R_{2k}]_{\underline{s}}^{\bar{s}} - \frac{k}{m - k}\,[S_{2m}\,R_{2k-1}]_{\underline{s}}^{\bar{s}}$$

for $k \lesseqgtr m$,

(90)

$$\int_{\underline{s}}^{\bar{s}} S_{2m+1}\, R_{2k+1} \cdot ds = \frac{1}{2\,(m-k)}\left[S_{2m+1}\, R_{2k}\right]_{\underline{s}}^{\bar{s}} - \frac{1}{2\,(m-k)}\left[S_{2m}\, R_{2k+1}\right]_{\underline{s}}^{\bar{s}}$$

for $k \lesseqgtr m$, and

(91)

$$\int_{\underline{s}}^{\bar{s}} S_{2m} R_{2k+1} \cdot ds = \frac{2\,m}{2\,m - 2\,k - 1}\left[S_{2m-1} R_{2k+1}\right]_{\underline{s}}^{\bar{s}} + \frac{1}{2\,m - 2\,k - 1}\left[S_{2m} R_{2k}\right]_{\underline{s}}^{\bar{s}}.$$

Also, from (83), (87), and (88) follow

$$\frac{\partial}{\partial s}\left(S_{2m}\, R_{2m-1}\right) = S_{2m}\, R_{2m} - 2\,m \cdot S_{2m-1}\, R_{2m-1}\,,$$

$$\frac{\partial}{\partial s}\left(S_{2m+1}\, R_{2m}\right) = -\,(2\,m + 1) \cdot S_{2m+1}\, R_{2m+1} + S_{2m}\, R_{2m}\,,$$

so that

$$(92) \qquad \int_{\underline{s}}^{\bar{s}} S_{2m}\, R_{2m} \cdot ds = \left[S_{2m}\, R_{2m-1}\right)_{\underline{s}}^{\bar{s}} + 2\,m \cdot \int_{\underline{s}}^{\bar{s}} S_{2m-1}\, R_{2m\ \ 1} \cdot ds,$$

$$(93) \qquad \int_{\underline{s}}^{\bar{s}} S_{2m+1} R_{2m+1} \cdot ds = -\frac{1}{2m+1}\left[S_{2m+1}\, R_{2m}\right]_{\underline{s}}^{\bar{s}} + \frac{1}{2m+1}\int_{\underline{s}}^{\bar{s}} S_{2m}\, R_{2m} \cdot ds.$$

By alternate applications of (92) and (93) any integral of the type $\int_{\underline{s}}^{\bar{s}} S_m\, R_m \cdot ds$ can be reduced to a linear function of expressions of the type $\left[S_k\, R_{k-1}\right]_{\underline{s}}^{\bar{s}}$ for values of k descending from m to 1 and of the integral

$$(94) \qquad \int_{\underline{s}}^{\bar{s}} S_0 R_0 \cdot ds = \int_{\underline{s}}^{\bar{s}} e^{-\frac{s^2}{2}} \cdot ds.$$

X. Case in which all Values of s are possible.

For $\underline{s} = -\,\infty$ and $\bar{s} = +\,\infty$ (that is, if *all* values of the residual are possible), the integrals in question have simple numerical expressions. Namely, whatever integer k may be,

$$s^k\, e^{-\frac{s^2}{2}} = 0 \qquad \text{for } s = \pm\,\infty\,;$$

therefore,

(95) $$R_k = 0 \qquad \text{for } s = \pm \infty\,;$$

therefore, equations (81) are all satisfied, by (89)-(91). Also, (92) and (93) become, for this case,

$$\int_{-\infty}^{+\infty} S_{2m}\, R_{2m} \cdot ds = 2\,m \int_{-\infty}^{+\infty} S_{2m-1}\, R_{2m-1} \cdot ds,$$

$$\int_{-\infty}^{+\infty} S_{2m+1}\, R_{2m+1} \cdot ds = \frac{1}{2\,m+1} \int_{-\infty}^{+\infty} S_{2m}\, R_{2m} \cdot ds\,;$$

by alternate applications of these two formulae, we obtain

$$\int_{-\infty}^{+\infty} S_{2m}\, R_{2m} \cdot ds = \frac{2\,m}{2\,m-1} \int_{-\infty}^{+\infty} S_{2m-2}\, R_{2m-2} \cdot ds,$$

$$\int_{-\infty}^{+\infty} S_{2m+1}\, R_{2m+1} \cdot ds = \frac{2\,m}{2\,m+1} \int_{-\infty}^{+\infty} S_{2m-1}\, R_{2m-1} \cdot ds\,;$$

and again, by repetitions of these,

$$\int_{-\infty}^{+\infty} S_{2m}\, R_{2m} \cdot ds = \frac{m\,!}{(m-\frac{1}{2})^{(m)}} \int_{-\infty}^{+\infty} S_0\, R_0 \cdot ds,$$

$$\int_{-\infty}^{+\infty} S_{2m+1}\, R_{2m+1} \cdot ds = \frac{m\,!}{(m+\frac{1}{2})^{(m)}} \int_{-\infty}^{+\infty} S_1\, R_1 \cdot ds.$$

Now, by (93), (63), and (68),

$$\int_{-\infty}^{+\infty} S_1\, R_1 \cdot ds = \int_{-\infty}^{+\infty} S_0\, R_0 \cdot ds = \int_{-\infty}^{+\infty} e^{-\frac{s^2}{2}} \cdot ds = \sqrt{2\,\pi},$$

where π represents the ratio of the circumference of a circle to its diameter; therefore,

(96) $$\int_{-\infty}^{+\infty} S_{2m}\, R_{2m} \cdot ds = \frac{m\,!}{(m-\frac{1}{2})^{(m)}} \sqrt{2\,\pi},$$

(97) $$\int_{-\infty}^{+\infty} S_{2m+1}\, R_{2m+1} \cdot ds = \frac{m\,!}{(m+\frac{1}{2})^{(m)}} \sqrt{2\,\pi},$$

so that, by (80),

$$(98)\qquad C_{2m} = \frac{1}{\sqrt{2\pi}}\,\frac{(m-\frac{1}{2})^{(m)}}{m\,!} \text{ for } 0 \leqq m \leqq \alpha \text{ (excepting } m = 1)$$

and

$$(99)\qquad C_{2m+1} = \frac{1}{\sqrt{2\pi}}\,\frac{(m+\frac{1}{2})^{(m)}}{m\,!} \text{ for } 1 \leqq m \leqq \beta.$$

It will be observed that C_{2m} is less and C_{2m+1} is greater than $\frac{1}{\sqrt{2\pi}}$; for very large values of m we have, by Stirling's formula,

$$(100)\qquad C_{2m} = \frac{1}{\pi\sqrt{m}}, \qquad C_{2m+1} = \frac{\sqrt{2m}}{\pi},$$

so that C_{2m} diminishes and C_{2m+1} increases without limit as m increases without limit; this fact will, perhaps, to a certain extent offset the conclusion that might be drawn from (61) that the values of the P's with odd suffixes are likely to be less than those of the P's with nearly the same even suffixes (on account of the presence of the odd functions S under the integral sign in the former and of the even functions S in the latter) in estimating the relative importance of the terms of $f(s)$ as given by (78) that contain R's with odd suffixes and R's with even suffixes.

For the C's with the smaller suffixes we have

$$(101)\qquad \begin{cases} C_0 = \dfrac{1}{\sqrt{2\pi}}, & C_4 = \dfrac{3}{8\sqrt{2\pi}}, \quad C_6 = \dfrac{5}{16\sqrt{2\pi}}, \text{ etc.,} \\ C_3 = \dfrac{3}{2\sqrt{2\pi}}, & C_5 = \dfrac{15}{8\sqrt{2\pi}}, \quad C_7 = \dfrac{35}{16\sqrt{2\pi}}, \text{ etc.,} \end{cases}$$

so that, by (27), (78), (68), (51), (63), and (101),

$$(102)\qquad \begin{cases} \psi(\xi) = \dfrac{1}{\mu} f(s) = \dfrac{1}{\mu\sqrt{2\pi}}\left[1 - \frac{3}{8}(1 - \frac{1}{3}\rho_4)(1 - 2s^2 + \frac{1}{3}s^4)\right. \\ - \frac{5}{8}(1 - \frac{1}{2}\rho_4 + \frac{1}{30}\rho_6)(1 - 3s^2 + s^4 - \frac{1}{15}s^6) + \text{etc.} \\ \left. - \frac{1}{2}\rho_3(s - \frac{1}{3}s^3) - \frac{5}{4}(\rho_3 - \frac{1}{10}\rho_5)(s - \frac{2}{3}s^3 + \frac{1}{15}s^5) + \text{etc.}\right]e^{-\frac{s^2}{2}}. \end{cases}$$

XI. Application to any Particular Case.

It seems reasonable to suppose that, in any case (whatever the values of $\underline{s}$ and $\overline{s}$) $f(s)$ will have small values for values of s beyond the limits of possible values, so that the extension of the limits to $-\infty$ and $+\infty$ will not make much difference in the values of the integrals by which the C's are determined; we may, then, take the values given by (98) and (99) as *approximate* in all cases.

The restriction of the development (78) of $f(s)$ to a finite number of terms is justified only by the fact that the P's that would occur in later terms are all negligible, that is that

$$(103) \qquad \begin{cases} P_{2m} = 0 \quad \text{for} \quad \alpha < m, \\ P_{2m+1} = 0 \quad \text{for} \quad \beta < m, \end{cases}$$

to the degree of approximation to which the calculations are restricted. In the special case for which $\underline{s} = -\infty$ and $\bar{s} = +\infty$ these equations (103) are *exactly* satisfied by (89)–(91), because the suffixes of the P's in (103) are different from those of any of the P's that occur in the right members of (79).

In applying the general method to any particular set of observations we have, then, to calculate, by (1)–(7) and (48) (from the observations), the values of z_0, the residuals ($\xi = z - z_0$), the p's, the ρ's, and the P's with so many successive suffixes as to satisfy ourselves that the P's with suffixes greater than a certain number are all negligible; the greatest even suffix and the greatest odd suffix of the P's that are not negligible are the values of $2\,\alpha$ and $2\,\beta + 1$, respectively; the values of the P's that are not negligible and the approximate values of the corresponding C's given by (98) and (99) being substituted in (77) determine an approximate form of $\theta(s)$; the positive and negative roots of $\theta(s) = 0$ that have the smallest absolute values are approximate values of $\underline{s}$ and $\bar{s}$, with which more accurate values of the C's can be calculated by (80); substituting these values of the C's in (77) we obtain a more exact form of $\theta(s)$, from $\theta(s) = 0$ more exact values of $\underline{s}$ and $\bar{s}$, and from (80) more exact values of the C's; and so we continue until the variations of the values of the C's are reduced to negligible quantities; the final values of the C's being substituted in (78) give the most exact form of $f(s)$ and, by (27), the most exact form of $\psi(\xi)$ that can be derived from the observations. The closeness with which the conditions (81) are satisfied by the final values of $\underline{s}$ and $\bar{s}$ may be regarded as a test of the *goodness of the observations*, that is of the closeness with which the residuals follow the law that they follow most closely.

It may be observed that there is no case for which $\alpha = 1$; more properly, the cases for which $\alpha = 1$ coincide with the cases for which $\alpha = 0$.

If $P_{2m+1} = 0$ for $1 \leqq m$, $f(s)$ as given by (78) is an even function of s and the relative frequencies of positive and negative residuals that have the same absolute magnitudes are equal, that is, *positive and negative residuals are equally probable.*

The calculation of the true values of the C's by (80). (92), and (93) will be facilitated by tables of the values of $S_m\, R_{m-1}$ and of the integral (94). The values of $S_m\, R_{m-1}$ will be required for such values of m as are suffixes of the P's that are not negligible, which will be small, undoubtedly, in all actual cases; tables of $S_m\, R_{m-1}$ for $m = 3, 4, 5, 6$ will probably be sufficient. For the values of the integral (94) the ordinary tables of

$$(104) \qquad \Phi\,(\gamma) = \frac{2}{\sqrt{\pi}} \int\limits_0^{\gamma} e^{-t^2} \cdot dt$$

can be used; namely,

$$\int\limits_0^{\gamma} e^{-\frac{s^2}{2}} \cdot ds = \sqrt{2} \int\limits_0^{\frac{\gamma}{\sqrt{2}}} e^{-t^2} \cdot dt = \sqrt{\frac{\pi}{2}} \cdot \Phi\!\left(\frac{\gamma}{\sqrt{2}}\right),$$

so that, observing that $\Phi\,(-\gamma) = -\,\Phi\,(\gamma)$, we have

$$(105) \int\limits_{\underline{s}}^{\bar{s}} e^{-\frac{s^2}{2}} \cdot ds = \sqrt{\frac{\pi}{2}} \cdot \left[\Phi\left(\frac{\bar{s}}{\sqrt{2}}\right) - \Phi\left(\frac{\underline{s}}{\sqrt{2}}\right)\right]$$

$$= \sqrt{\frac{\pi}{2}} \cdot \left[\Phi\left(\frac{\bar{s}}{\sqrt{2}}\right) + \Phi\left(\frac{-\underline{s}}{\sqrt{2}}\right)\right],$$

where $\bar{s}$ and $-\,\underline{s}$ are both positive.

XII. Probable Values.

We have defined the *probable value* of the observed quantity to be that value that is just as likely to be exceeded as not; if the *probable residual* ξ_p and the *probable relative residual* s_p be taken to correspond to this probable value z_p, ξ_p and s_p will be determined by the equation

$$(106) \qquad \int\limits_{\underline{\xi}}^{\xi_p} \psi\,(\xi) \cdot d\xi = \int\limits_{\underline{s}}^{s_p} f(s) \cdot ds = \tfrac{1}{2}$$

and z_p by

$$(107) \qquad z_p = z_0 + \xi_p,$$

in accordance with (3), so that the arithmetical mean is (in general) not the probable value even of a quantity directly observed (as it is assumed to be in the method of Gauss).

XIII. The Ordinary Case.

In the ordinary theory it is assumed that

$$(108) \qquad P_k = 0 \quad \text{for} \quad 3 \leqq k,$$

so that the only P regarded as not negligible is $P_0 = 1$, which being substituted in (77) with the approximate value $C_0 = \frac{1}{\sqrt{2\,\pi}}$, by (101), gives, by (51) and (63),

$$\theta(s) = C_0\, P_0\, S_0 = \frac{1}{\sqrt{2\,\pi}};$$

$\theta(s) = 0$ has, then, no real finite root, so that

$$\underline{s} = -\infty\,, \; \overline{s} = +\infty$$

and (80) gives again the same value of C_0 as (101), which is, therefore, the most exact value possible on the assumption made. We have further, by (78) and (27),

$$(109) \qquad f(s) = \frac{1}{\sqrt{2\,\pi}}\, e^{-\frac{s^2}{2}},\; \psi(\xi) = \frac{1}{\mu\sqrt{2\,\pi}}\, e^{-\frac{\xi^2}{2\mu^2}};$$

while (60) gives

$$\rho_k = \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{+\infty} s^k \cdot e^{-\frac{s^2}{2}} \cdot ds,$$

that is

$$(110) \qquad \rho_{2m+1} = 0 \quad \text{for} \quad 0 \leqq m$$

and

$$(111) \qquad \rho_{2m} = 2^m\,(m - \tfrac{1}{2})^{(m)} \quad \text{or} \quad p_{2m} = 2^m\,(m - \tfrac{1}{2})^{(m)} \cdot \mu^{2m}$$

for $0 \leqq m$, so that, far from being negligible for the greater values of m, ρ_{2m} increases beyond all limit as m increases.

XIV. Two Other Particular Cases.

For the purpose of illustrating the variety of forms the general theory may take in particular cases, and especially its possible deviation from the ordinary theory, we have plotted the curve represented by $y = \frac{1}{\mu} f(s)$ (the *probability-curve*) for $\alpha = 0$, $\beta = 1$ and for $\alpha = 2$, $\beta = 0$, using the approximate values of C_0, C_3, and C_4 given by (101), for such different values of ρ_3 and ρ_4 (in the two cases, respectively) as to show all the

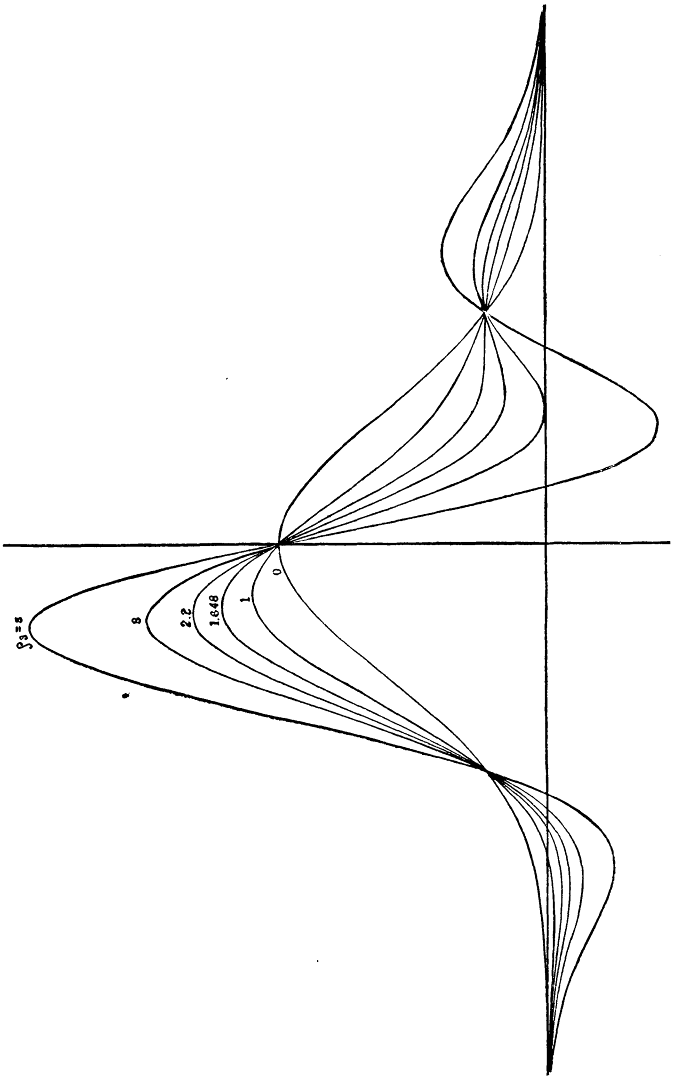

FIGURE 1.

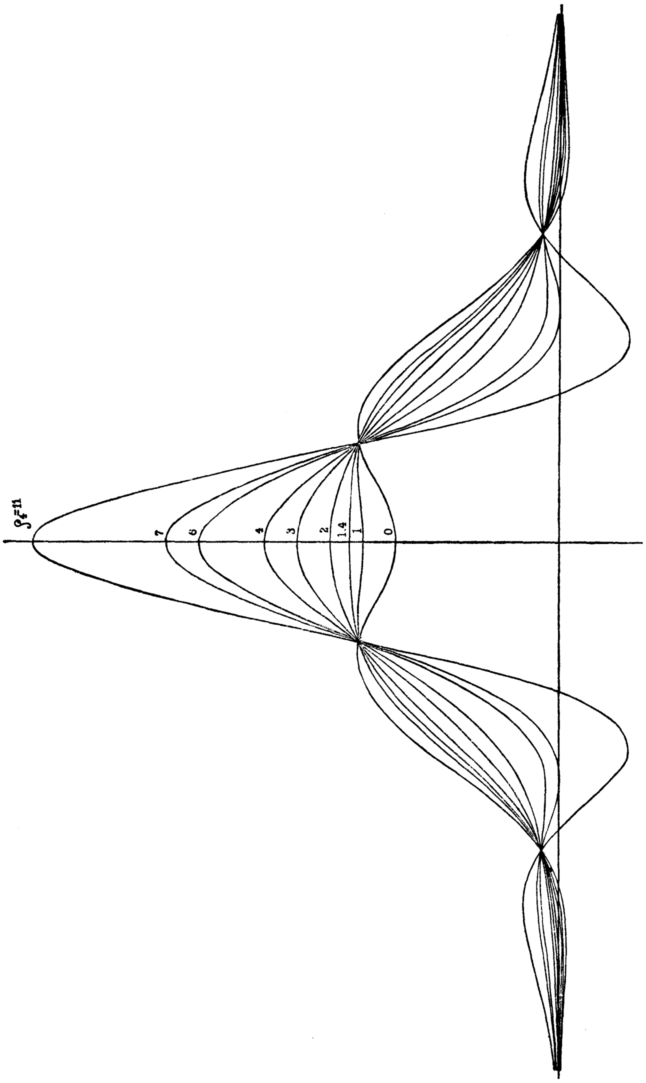

FIGURE 2.

essentially different forms for these pairs of values of α and β. Figure 1 shows the locus of the equation

$$y = \frac{1}{\mu \sqrt{2\pi}} \left[1 - \tfrac{1}{2} \rho_3 \left(s - \tfrac{1}{3} s^3\right)\right] e^{-\frac{s^2}{2}} \tag{112}$$

for $\rho_3 = 0, 1, 1.648, 2.2, 3, 5$, and Figure 2 the locus of the equation

$$y = \frac{1}{\mu \sqrt{2\pi}} \left[1 - \tfrac{3}{8} \left(1 - \tfrac{1}{3} \rho_4\right) \left(1 - 2 s^2 + \tfrac{1}{3} s^4\right)\right] e^{-\frac{s^2}{2}} \tag{113}$$

for $\rho_4 = 0, 1, 1.4, 2, 3, 4, 6, 7, 11$. The centre of each figure corresponds to $s = 0$ and the two extremes to $s = \pm 4$; the horizontal line is the axis of abscissæ. It will be seen that the ordinates of all the curves are practically 0 for values of s whose absolute magnitudes exceed 4. The curve for which $\rho_3 = 0$ in Figure 1 and that for which $\rho_4 = 3$ in Figure 2 coincide with the ordinary probability-curve. The curve for which $\rho_3 = 3$ in Figure 1 and that for which $\rho_4 = 7$ in Figure 2 touch the axis of abscissae; these values are, by the footnote on page 173, the extreme values of ρ_3 and ρ_4, respectively; each figure contains one curve that corresponds to a value beyond these limits, which shows the discontinuity of $\underline{s}$ or $\bar{s}$ as the curve passes through the form that has contact with the axis. While ρ_4 is essentially positive, as a mean even power of the relative residual, ρ_3 may be negative; if the sign of the value of ρ_3 that corresponds to either curve in Figure 1 is changed, the curve will be reversed, right and left. In the cases represented in Figure 1, the maximum relative frequency and the probable value of the observed quantity correspond to negative values of s; in the cases represented in Figure 2, the maximum relative frequency corresponds to $s = 0$ for some values of ρ_4, while there are two maxima for the smaller values of ρ_4, the probable value of the observed quantity corresponding always to $s = 0$.

Because $s = 1$ corresponds to $\xi = \mu$, that is to $\xi^2 = p_2$, where p_2 is the mean square of the residual, it is evident that at least one of the limits $\underline{s}$ and $\bar{s}$ must be considerably greater than 1 in absolute magnitude, in any case; but the curves plotted show that very little of accuracy is gained by considering values of s beyond $\pm$ 4 in the cases represented by them.

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